III

MODERN DIFFERENTIAL GEOMETRY

THE classical differential geometry is a branch of Euclidean geometry, namely, the local theory of curves and surfaces which have analytical equations in terms of Cartesian coordinate systems. It was called differential geometry because it employs the results and methods of the differential calculus. During the past three decades there have developed two other differential geometries, namely, the Affine and Projective, in which the circumambient space has been assumed to be affine or projective instead of Euclidean. Indeed, there have also been some beginnings of a conformal differential geometry, a theory of curves and surfaces in a space whose group is the group of transformations by reciprocal radii.

It is not however chiefly from these generalizations of classical differential geometry that has come the efflorescence of new geometric ideas that I wish to report on today, but rather from an extension of the classical theory of deformation of surfaces. You will remember that there is talk in differential geometry of two differential forms, the first fundamental form

$$Edu^2+2Fdudv+Gdv^2$$

and the second fundamental form

$$Ddu^2+2D'dudv+D''dv^2.$$

The surface in question is supposed to be given in "parametric form" by equations

$$x = x(u,v), \ y = y(u,v), \ z = z(u,v)$$

where x, y, z are rectangular Cartesian coordinates and u and v are arbitrary parameters. The coefficients $E, F, G,$ and D, D', D'' are functions of u and v and completely determine the local properties of the surface.

The last phrase means that if two sufficiently small fragments of surface are given, to each of which parameters u, v can be assigned so that the six coefficients, $E, \ldots, D''$ are the same for both, then the one fragment can be rigidly displaced so as to fit on the other. If we drop D, D', D'' out of consideration and attend only to E, F, and G, we find that when two fragments of surface can be parameterized so that they have the same E, F, and G, the one can be "deformed" so as to fit on the other. The word "deform" here implies that the surface is thought of as a sort of non-extensible membrane which can be twisted and otherwise changed in shape in any manner which keeps unchanged the distance between any two points as measured along a curve in the membrane itself. In fact the formula for the length of any curve lying on the surface is

$$(1) \qquad \int \sqrt{Edu^2 + 2Fdudv + Gdv^2}.$$

The Cartesian coordinates x, y, z have disappeared from view. Everything depends on E, F, and G, which are functions of the parameters u and v, and so all our business is transacted on the surface itself without ever looking outside. It is the intrinsic properties of the surface with which we are dealing. The surface might quite well be spread out in a Euclidean space of four rather than three dimensions, and, indeed, there is no occasion for requiring it to be immersed in any ambient space. The surface is itself the space of our theory, and u and v are coordinates of this space in precisely the sense that we have been speaking of coordinates in the two previous lectures.

The generalization of this doctrine was made by Riemann in his Habilitationsschrift (1854). Consider a space the points of which can be denoted by sets of n real coordinates $(x^1, \ldots, x^n)$ and in which the length of any curve is given by the integral of ds where

$$ds^2 = \sum_{i,j} g_{ij} dx^i dx^j \quad (i, j = 1, 2, \ldots, n)$$

the coefficients g_{ij} being functions of the coordinates. Such a space we now call a Riemannian space.

In case the coordinates $x^1, \ldots, x^n$ can be so chosen that the quadratic form for ds reduces to

$$(dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$$

the space reduces to a Euclidean space, at least locally. In general, no such simplifying choice of coordinates is possible for an n-cell of the space, no matter how small the n-cell. Nevertheless the space has a very full set of properties. For example, there is a system of curves analogous to the straight lines. These are the geodesics, or curves of extreme length. In general, however, there are no surfaces analogous to the Euclidean planes. The existence in a Riemannian space of a given class of figures, such as planes, which are more or less analogous to some class of figures studied in Euclidean geometry, will depend upon the coefficients g_{ij} of the fundamental quadratic form satisfying some particular set of relations.

The theory of a particular Riemannian space is a full-fledged geometry. What is ordinarily meant by Riemannian geometry, however, is the theory of Riemannian spaces in general, firstly the theorem which hold for all Riemannian spaces, and secondly the properties of classes of Riemannian spaces obtained by imposing restrictions on the definition of distance. The Riemannian geometry is identical with the theory of quadratic differential forms, for everything depends on the formula for ds^2.

The Riemannian geometry as it exists today is for the most part, analytically speaking, formal, or geometrically speaking, local. This means that a typical theorem of the science as it exists today relates only to the neighborhood of a single point. The sort of Riemannian geometry which I think should and will be developed in the near future was touched on in the lecture yesterday.

The carrier space, the playground of the theory, is a regular manifold of n dimensions. This manifold is described by means of the axioms regarding "allowable" coordinate systems to which I referred yesterday. Each allowable coordinate system is a correspondence between a fragment of the regular manifold (perhaps just an n-cell) and a set of sets of numbers $(x^1, \ldots, x^n)$. These sets of n coordinates we think of as arithmetic points in the arithmetic space of n dimensions.

In order that the formula for distance (1) can be written in any coordinate system, the functions g_{ij} must be known in all coordinate systems. Indeed, specifying a Riemannian space amounts exactly to specifying the functions g_{ij} for all coordinate systems. Thus a full set of axioms for a particular Riemannian space would be (1) the general axioms for a regular manifold of the right dimensionality, (2) the axioms specifying the g_{ij}'s in all allowable coordinate systems, (3) whatever additional axioms are requisite to fix the topological character of the space.

With regard to (2) the specification of the g's must merely satisfy the requirement that the g's in different coordinate systems whose domains overlap be related by the correct tensor "law of transformation." This is a well-known technical detail.

As an example of what is meant by (3) we might demand that the regular manifold be closed and that the topological constants (Betti numbers, etc.) take on particular values. But it may very well happen that for a specific choice of the g's certain values of these constants are excluded.

Thus, for example, suppose we require that the g's shall be such that in any small domain, coordinates may be chosen so that the fundamental quadratic form is a sum of squares of the differentials. The Riemannian space under this condition is said to be "locally Euclidean," because in the neighborhood of each point it is indistinguishable from the ordinary Euclidean space of n-dimensions. In the two-dimensional case it is not possible that a locally Euclidean space should be homeomorphic with a sphere, though it may be homeomorphic with an anchor ring.

The latter possibility was discovered by W. K. Clifford (1873) who found a locus, now known as the Clifford surface, in a three-dimensional elliptic space the geometry upon which is locally Euclidean. Generalizing from his, we have the problem: what can be said about the topology of a locally flat n-dimensional Riemannian space? Or still more generally, what about the topology of Riemannian spaces of constant curvature? This last is known as the problem of the Clifford-Klein space forms. It was formulated by Klein in 1890 in an article in the *Mathematische Annalen* in which he drew attention to the significance of Clifford's discovery. The problem was studied extensively by Killing in a book called *Grundlagen der Geometrie* (1893) and again with the aid of modern topological ideas by H. Hopf.[1] The essential step in its solution is to observe that any Clifford-Klein space has a universal covering space which is either a Euclidean space or a sphere or a hyperbolic non-Euclidean space. This carries the problem back to the determination of discontinuous groups of motions without fixed points in these covering spaces.

The Clifford-Klein problem is only one of a very important class of problems: assuming that a Riemannian space satisfies certain local conditions at every point, what can we say about its topology? As local condition we could require for instance that the covariant derivative of the curvature tensor should vanish, or that the g's should satisfy Einstein's differential equations for gravitation potentials, and so on.

Clearly there are plenty of outstanding problems in the Riemann geometry if we take the topology of the space into account. Moreover the type of problem I have been indicating is by no means the principal one. One has only to remark that the history of a dynamical system is represented by a geodesic in a Riemannian space and then to recall the researches of Poincaré and Birkhoff on these systems of curves, or, going a bit further, those of Morse on global calculus of variations.

I have dwelt on this borderland field between differential geometry and topology because I am much impressed with its importance, but the main subject of this lecture was to be the generalizations of the Riemannian geometry which have been a feature of the last decade of mathematical research. The definition of a Riemannian space was so stated that the generalization should be obvious. We have (1) a regular manifold of n dimensions with its allowable coordi-

[1] *Math. Ann.*, Vol. 95, 1925.

nate systems and (2) a rule for finding the length of any analytic curve. When only the specification (1) has been made, our space is thoroughly empty and amorphous, but when (2) has been specified we have something that may be described as structure. We can talk about the distance between two points and we can, as it were, find our way about by traveling along geodesics. The generalization from Riemannian geometry consists in replacing (2) by some other type of "structure."

One of the most important classes of generalized spaces obtainable by this process is the class of spaces of paths. By a system of paths we will mean a set of curves in a regular manifold which are such that, locally speaking, any two points are joined by one and only one curve of the set. That is to say, any point of the regular manifold is contained in an n-cell such that two points of the n-cell are joined by one and only one path. The regular manifold with its paths is a space of paths. The paths are obviously a generalization of the straight lines of a Euclidean space and of the geodesics of a Riemannian space. They serve the same purpose within the manifold, that of helping us to find our way about.

On this general basis the geometry of paths is still but little developed. In order to treat it by the standard methods of differential geometry one must make assumptions about the way in which the paths are represented by equations. This leads to a number of subcases which have been surveyed in an interesting manner by J. Douglas in the *Annals of Mathematics*,[1] the principle of classification being the ways in which the paths may be parameterized, i.e., expressed in the form

$$x^i = f^i(t).$$

[1] Vol. 29, 1928.

The most cultivated part of the subject is the affine geometry of paths. This is the theory of the class of spaces of paths in which it is possible to represent the paths by differential equations of the form,

$$\frac{d^2x^i}{dt^2}+\sum_{j,k}\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0. \tag{2}$$

The coefficients Γ^i_{jk} are functions of $(x^1, \ldots, x^n)$, and are known as components of an "affine connection."

The affine connection itself is an abstract thing whose sole reason for existence is to have components. For if we change to another coordinate system, the paths will be represented by another set of differential equations of the same form as (2) but with different functions Γ^i_{jk}. Thus we have a set of components Γ^i_{jk} in each allowable coordinate system, and whenever the domains of two allowable coordinate systems overlap there is a definite formula for transforming the Γ's of one system into those of the other.

An affine connection is a special case of a "geometric object with components." A still simpler case of such a geometric object is afforded by a Riemannian geometry. The functions g_{ij} are uniquely determined for every allowable coordinate system. They are the components of a geometric object called a tensor of the second order.

This idea of a geometric object with components may be used to characterize an extremely wide class of generalized geometries. *The theory of any geometric object with components in each allowable coordinate system of a regular manifold is a geometry.* In case the geometric object is a tensor of the second order, the geometry is Riemannian. In case the geometric object is an affine connection, the geometry is an affine geometry of paths.

The affine geometry of paths has been the field of extensive and brilliant researches which I must pass over, mentioning only the names of Cartan, Eisenhart, Schouten, Thomas, Weyl, to quote only a few leading ones. Among the ideas which have taken form as a result of these researches, is that of a tangent space to the regular manifold or, as we shall sometimes call it, the underlying space.

A primitive form of the idea is, for example, the conception that in the neighborhood of a point a Riemannian space is approximately Euclidean. Thus we can think of a Riemannian space being made up (using the language of Cartan) of a multitude of little facets each of which is a Euclidean space. To get something explicit in place of this intuition we go back to an observation the importance of which was recognized by Sophus Lie, that when the coordinates $x^1, \ldots, x^n$ undergo an arbitrary analytic transformation,

$$\bar{x}^i = \bar{x}^i(x), \tag{3}$$

their differentials at any point undergo only a linear homogeneous transformation,

$$d\bar{x}^i = \sum_j \frac{\partial \bar{x}^i}{\partial x^j} dx^j. \tag{4}$$

Let us interpret the dx's as coordinates in a space T of n dimensions. The sets of differentials $dx^1, \ldots, dx^n$ associated with a given point of the underlying space are perfectly arbitrary. Therefore the space T is like an ordinary affine (or Euclidean) space. It has a family of specially significant coordinate systems which are related by the centered affine group. Therefore it is a centered affine space. I say "centered" because there is one point, that for which all the differentials are zero, which plays a special rôle. This point we call the "point of contact" of T, and identify with the point of the underlying space at which the differentials are taken.

There is one of these tangent spaces associated with each point of the underlying space, and the tangent spaces are all distinct. No two of them have a point in common. When we transform coordinates by (3) in the underlying space, the Cartesian coordinates in all the tangent spaces simultaneously undergo the transformations (4). This all holds good for an arbitrary regular manifold.

Now if the regular manifold is the bearer of a Riemannian geometry, there is a Euclidean geometry automatically determined in each tangent space, namely, that for which the distance from the contact point to any point $(dx^1, \ldots, dx^n)$ is given by

$$\sqrt{\sum_{i,j} g_{ij} dx^i dx^j},$$

for the g's are constants relative to a fixed tangent space. They depend only on the coordinates $x^1, \ldots, x^n$ of the point of contact. The Riemannian geometry is in a quite precise way the theory of all these Euclidean spaces grouped together by their contacts with the underlying space.

Again, if the regular manifold is the bearer of an affine connection, the formula

$$\frac{dX^i}{dt} + \sum_{j,k} \Gamma^i_{jk} X^j \frac{dx^k}{dt} = 0 \tag{5}$$

determines what is called an *affine displacement.* Supposing that

$$x^i = x^i(t) \tag{6}$$

is a curve, the ordinary differential equations have solutions of the form

$$X^i(t) = \sum_j A^i_j(t) X^j_0 \tag{7}$$

where X^j_0 are arbitrary constants. We may interpret $(X^1_0, \ldots, X^n_0)$ as coordinates of a point in the tangent space at the point $t=0$ of the curve (6) and $X^i(t)$ as coordinates of a point in the tangent space at a point t of the same curve. The equations (7) then represent an affine transformation from the tangent space at $t=0$ to the tangent space at the variable point t.

Thus whenever an affine connection is specified, a curve joining two points of the underlying space determines a unique affine transformation of the tangent space at the first point into the tangent space at the second point. The generalized affine geometry can thus be thought of as the theory of the totality of affine tangent spaces together with the relations among them represented by these affine transformations.

We still have an affine displacement if the formula (5) be replaced by

$$\frac{dX^i}{dt}+\sum_{j,k}\Gamma^i_{jk}X^j\,\frac{dx^k}{dt}+\sum_j A^i_j\frac{dx^j}{dt}=0 \tag{8}$$

where A^i_j are components of an arbitrary "mixed tensor." But whereas the displacements determined by (5) carry points of contact into points of contact, this is not the case for (8).

In case (5) is replaced by

$$\frac{dX^i}{dt}+\sum_{j,k}\Gamma^i_{jk}X^j\,\frac{dx^k}{dt}+X^i\sum_{j,k}B_{jk}X^j\frac{dx^k}{dt}+\sum_k A^i_k\frac{dx^k}{dt}=0 \tag{9}$$

the coefficients of which now satisfy a somewhat different law of transformation from that of (8), a discussion analogous to that outlined above for (5) will show that the displacement defined is in general a non-affine projective one. We now interpret the $X^1,\dots,X^n$ as coordinates in a set of tangent projective spaces, one for each point of the underlying space. The theory of this totality of projective spaces as they are joined together by the underlying space and the projective displacements connecting them is a generalized projective geometry. There is a system of paths in the underlying space, definitely related to the projective displacements so that this geometry can also be regarded as a geometry of paths.

A similar process will define a generalized conformal geometry. Indeed any geometry in the sense of Klein can be generalized in this manner. Introduce a space of the Klein type as a tangent space at each point of the underlying manifold and a displacement formula analogous to (5) which will define an isomorphic displacement of the tangent space at any point, along any curve, to any other point.

The generalized spaces thus defined constitute a close generalization of the Klein spaces. Although such a generalized space is not in general left invariant by a group of transformations, there is a group which plays a rôle in characterizing these spaces.

This class of generalized spaces, which was first characterized by E. Cartan, seems to me an exceedingly elegant one. I cannot, however, go as far in admiration of it as some mathematicians who are willing to accept it as the last word in the definition of a geometry. It seems to me that it is simply one very elegant way of imposing a structure upon a generalized manifold. In practice it works out as a way of defining a geometric object with components. The concept of a geometric object with components is more general than that of a displacement, and there seems to me to be good reason to regard the theory of any such object as a geometry. Furthermore, it is not merely by the use of such objects that one may impose structure upon a regular manifold. So I come back to the point of view that a geometry is the theory of a regular manifold with structure. I don't think that any less general concept of a geometry can survive for long.

Another point which may well be made in this connection is that there may be several displacements which play rôles in the theory of a single geometric object, and it seems highly undesirable to say in such a case that we have several distinct geometries. For we obviously have a single mathematical theory.

A case of this sort is the projective relativity.[1] Here the fundamental geometric object determines a non-degenerate quadric in each tangent space and there is a family of projective displacements which carry these quadrics into themselves. One or another of these displacements may be chosen for physical interpretation, but they are all component parts of the geometry in question.

In conclusion I should like to say a few words about a new class of geometric objects which I think will soon become important, namely, the spinors.[2] These have gradually emerged from the study of the invariant theory of Dirac's differential equation for the electron. This invariant theory, to which von Neumann as well as Dirac himself contributed, was on Ehrenfest's suggestion put into a form which introduced the spinors as definite objects, by van der Waerden. It was generalized from special to general relativity by Weyl, who spoke on the subject here in Houston a couple of years ago, and by Fock; and recently the work of Weyl and Fock has been extended by Schouten who lectured on the subject a year ago at Cambridge and Princeton.

What I should like to do is to point out a bit of relevant geometry which, I think, has not yet been properly understood by those who are working on the subject. The relativity theory has to do with a Riemannian geometry of four

[1] See the booklet on *Projektive Relativitätstheorie* written subsequently to the delivery of this lecture, and published in *Ergebnisse der Mathematik*, Berlin, 1933.

[2] This part of the lecture was given in greater detail in a supplementary talk before Professor Wilson's seminar.

dimensions. In each tangent affine space there is a quadratic cone whose equation is

$$\sum_{i,j} g_{ij} dx^i dx^j = 0.$$

This cone is a system of straight lines through the contact point. If we take a 3-space S_3 not going through the origin (all this is happening in the real tangent 4-space) it will meet each line of the cone in a point, and this system of points will be an ordinary quadric in S_3. As you know, such a quadric contains two systems of "generating" straight lines which are in general imaginary. Hence the cone contains two systems of generating planes, the projections from the contact point of the two systems of straight lines. Any line of the cone is an intersection of a plane from one system with a plane from the other system.

We know that the lines of one system of generators of a quadric depend upon one complex parameter, or, since it is better to use a homogeneous representation, upon two complex parameters. The same remark applies to the planes of either system of generating planes of the cone. Let us call the parameters of the first system ψ^1 and ψ^2, and those of the second system ψ^3 and ψ^4. Then, as is well known to students of projective geometry, we can give a simple formula for the points on the quadric which is linear in ψ^1 and ψ^2 and also in ψ^3 and ψ^4.

This formula is left unaltered if ψ^1 and ψ^2 are replaced by any linear combinations of themselves

$$\text{(10)} \qquad \begin{aligned} \bar{\psi}^1 &= a\psi^1 + b\psi^2 \\ \bar{\psi}^2 &= c\psi^1 + d\psi^2 \end{aligned} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

and if ψ^3 and ψ^4 are replaced by linear combinations of themselves with coefficients which are complex conjugates of those appearing in (10). This corresponds to the fact that the parameter representation must depend on some sort of a frame of reference. In fact a system of generating planes of our cone is what we call in projective geometry a "one dimensional form" (like a pencil of points) and the parameters can be assigned arbitrarily to any three of its elements and then are determined for all the rest. This is exactly the freedom implied by the transformations (10).

The complex numbers ψ^1 and ψ^2 are the components of one spinor and ψ^3 and ψ^4 are the components of another spinor. The ratio of the components of the first spinor fixes a definite plane of one system of generating planes as soon as the frame of reference in this system of planes is fixed, and the second spinor has the same meaning with respect to the other family of generating planes.

The changes of the frames of reference for the two systems of generators have nothing to do with transformations of point coordinates either in the underlying space or in the tangent spaces. Thus a spinor of the sort we are talking about is a geometric object which has two components in any coordinate system and any transformation of coordinates leaves them unaltered. When, however, the frame of reference of the generating planes (remember that there is such a frame in each tangent space) is changed, the components of the spinor undergo a linear transformation. Moreover, the components of two spinors ψ^1, ψ^2 and ψ^3, ψ^4 undergo complex conjugate transformations.

When we make an affine displacement of tangent spaces of the sort that is determined by the Riemannian geometry, we necessarily displace the planes of a system of generators of the cone in one tangent space into the corresponding figure in another tangent space. This of course determines a definite displacement formula, analogous to (5) for the spinor ψ^1, ψ^2 and also one for ψ^3, ψ^4. These formulas are not at all complicated and I have in fact developed them in full in this way in my seminar in Princeton. They agree with the results of Weyl, Fock, and Schouten. There is thus determined a significant covariant differentiation of spinors. On setting down the simplest relation among these covariant derivatives which does not imply that all the covariant derivatives vanish, we obtain the differential equations of Dirac, generalized to the case of general relativity. The ordinary Dirac equation is of course what these reduce to when the quadratic differential form assumes the special relativity form.

From this point of view the invariance of the Dirac equation is evident. It is also evident that the Dirac equation is a mathematical consequence of the geometric structure implied by relativity. Therefore it would seem to me to be unavoidable if we accept the relativity theory. Conceivably the physical interpretation assigned to these differential equations by quantum theory is incorrect. But the equations are there as soon as the relativity theory is accepted, and presumably should have a physical meaning. It would seem that the success of the Dirac theory would be evidence in favor of relativity and its failure would leave an open problem which must be solved before the relativity theory could be fully accepted.

Mathematically, the theory of spinors seems to me to open up a very interesting vista of possibilities. Heretofore, the theory of tangent spaces has made very little use of the detailed geometry in these spaces. It has only used the general properties of the transformations between them. But the outline of the two-component spinor theory which I have just given, really employs the geometry of the quadric. I think we may look forward to the time when large bodies of the detailed theory of algebraic surfaces and other configurations will have to be used in describing what goes on in the tangent spaces of a regular manifold. The relations between these figures in the various tangent spaces will give rise to generalized theories of displacement and of covariant differentiation. The representation of systems of figures in each tangent space by means of parameters will give rise to new sorts of geometric objects with components of which the two-component spinors are perhaps the simplest.

I can give another example which actually occurred to me before the two-component spinor case. You will doubtless remember the Pluecker-Klein correspondence between the straight lines of a three-dimensional projective space P_3 and the points on a quadric Q_4 in a five-dimensional projective space P_5. Some of you surely remember what a flood of light this threw upon our ideas about line geometry.

What I propose now is to use this correspondence in the inverse sense to that intended by its discoverers. Suppose, for example, we are studying a four-dimensional generalized conformal geometry. The tangent spaces in this case are 4-spheres. Each has the internal structure of the Q_4 in P_5. Hence it is the image of the straight lines of a projective three space P_3. Thus we arrive at the idea of associating with each point of our underlying space a P_3 the lines of which image the points of the tangent space.

The points of P_3 are represented by four homogeneous coordinates $\psi_1, \psi_2, \psi_3, \psi_4$, each. The assignment of these coordinates has the arbitrariness which is habitual to a projective frame of reference, namely, that of an arbitrary linear homogeneous transformation. Thus we are led to what we will call four-component spinors. A geometric object of this sort has in any coordinate system four components which are functions of the coordinates. These components behave like scalars under transformations of coordinates, i.e. their values do not change. On the other hand, under transformations of the frame of reference in the P_3's they undergo linear transformations the coefficients of which are arbitrary functions of the coordinates.

It must be remembered that P_3 is a complex space and so are Q_4 and P_5. On the other hand, our tangent space is real. Therefore some means must be found for selecting a real subspace of P_5 in which the real Q_4 has the correct signature. The apparatus for doing this is at hand in Segre's theory of "chains," "anticollineations," "antiquadrics," etc., but I have not yet worked out the full details. However, the work has gone far enough to make it certain that this theory of Segre's which we used to admire but always felt was far to one side of the main current of mathematics, is going to have an application in modern differential geometry and physics.

I say physics, partly because it is possible that the relativity theory will need to use the conformal geometry, and in that case there is a Dirac equation in terms of the four-component spinors. Furthermore, there is an application of the four-component spinors to projective relativity. This may be seen geometrically as follows:

The projective relativity 4-space has four-dimensional real tangent spaces T_4 each containing a quadric Q_3 of a specific signature. Returning to the Pluecker-Klein correspondence we recognize that any flat space P_4 in P_5 meets the quadric Q_4 in a quadric Q'_3, and that the points of Q'_3 correspond to the straight lines of a linear complex C in P_3. To apply the four-component spinor theory to projective relativity we need to identify T_4 with a real subspace of the complex space P_4 in such a way that the quadric in which this real subspace meets Q_4 has the signature of Q'_3. This once more requires a reference to Segre's theory. We have to specify as invariant in P_3 not only the linear congruence C but also a properly chosen "anti-involution of the second kind."

Of course, I don't expect you to grasp all these details instantaneously, but I thought it well to mention a few of them in order to give concreteness to the prediction that the ideas of classical projective and algebraic geometry are coming full-force and full-blown into modern differential geometry, *via* the tangent spaces.

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